arXiv:hep-ph/9603351v2 4 May 1996

Gauge Invariance in the Process
$$e^+e^- \to \bar{\nu}_e e^- W^+ \to \bar{\nu}_e e^- u\bar{d}$$

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Abstract

The process $e^+e^- \to \bar{\nu}_e e^- W^+ \to \bar{\nu}_e e^- u\bar{d}$ is considered as an example of the problems associated with maintaining gauge invariance in matrix elements involving unstable particles. It is shown how to construct a matrix element that correctly treats width effects for the intermediate unstable W boson and that is both $SU(2)_L$ and $U(1)_{\text{e.m.}}$ gauge-invariant. $SU(2)_L$ gauge-invariance is maintained by Laurent expansion in kinematic invariants and $U(1)_{\text{e.m.}}$ gauge-invariance is enforced by means of a projection operator under which the exact matrix element is invariant.

1 Introduction

In recent years the difficulty and importance of producing exactly gauge-invariant amplitudes for processes involving unstable particles has come to be appreciated. Prior to 1991 all calculations of electroweak physics at the Z^0 resonance, and even the definition of M_Z, were gauge-dependent. A variety of techniques have been applied to try to remove explicit gauge-dependence. Some of these involve the generation of gauge-invariant Green's functions, such as self-energies and vertex corrections [1, 2, 3]. Such techniques however replace dependence on, say, the gauge parameter by some arbitrary choice in the definition of the Green's function. In the case of the so-called 'pinch technique' the procedure used may be rigorously justified by the background field method [4, 5] for a particular choice of the gauge parameter, ξ. The fact that a particular choice of gauge is involved indicates that the gauge dependence has been disguised but not eliminated. The appearance of gauge-dependent amplitudes is a symptom of a perturbative expansion that does not reflect the physics of the problem.

The only method to produce genuinely gauge-invariant amplitudes at all orders in perturbation theory is that of Laurent expansion as proposed in ref.[6, 7]. This expansion should be applied even in the case of amplitudes generated via the pinch technique. In ref.[8] it was pointed out that the expansion was not just a mathematical trick. The leading resonant term in the expansion represents the finite propagation and subsequent decay of a physical unstable particle. As such it constitutes a distinguishable physical process and therefore must be exactly gauge-invariant. This insight was used to give an answer to the long-standing problem of how to calculate production cross-sections for unstable particles and was applied to the process $e^+e^- \to Z^0 Z^0$.

The presence of an unstable particle is indicated by a finite propagation length separating production and decay vertices. The contributions responsible for finite propagation are just those that are resonant in the invariant mass of the decay products. Thus, for example, to calculate the cross-section $e^+e^- \to Z^0 Z^0$ one calculates the matrix element for the process $e^+e^- \to Z^0 Z^0 \to (f_1 \bar{f}_1)(f_2 \bar{f}_2)$ and extracts the part resonant in the invariant masses, p_1^2 and p_2^2, of the $(f_1 \bar{f}_1)$ and $(f_2 \bar{f}_2)$ pairs respectively. This is accomplished by means of a Laurent expansion in each of the variables p_1^2 and p_2^2. In the notation of ref.[9] the general matrix element for the process takes the form

$$\mathcal{M}(..., p_1, p_2, ...) = \sum_i l_i(..., p_1, p_2, ...) A_i(t, u, p_1^2, p_2^2) \tag{1}$$

where the l_i are 'standard covariants', with external wavefunctions attached, that form a basis for the spinor and Lorentz tensor structure of the matrix element. They therefore transform as $\mathcal{M}$ does. The A_i are Lorentz scalar functions of the independent Lorentz invariants of the problem and contain no kinematic singularities. The resonant part is extracted by taking the leading terms $\hat{A}_i$ in a Laurent expansion first in p_1^2 and then p_2^2.

The cross-section formed from the resonant parts

$$\widehat{\mathcal{M}}(...,p_1,p_2,...) = \sum_i l_i(...,p_1,p_2,...)\hat{A}_i(t,u,p_1^2,p_2^2) \qquad (2)$$

or in this case doubly resonant parts, of the matrix element and then summed over all possible final states constitutes the cross-section $\sigma(e^+e^- \to Z^0Z^0)$. If the standard covariants l_i do form a basis then the scalars A_i will be gauge-invariant and hence so will the $\hat{A}_i$.

The external particle wave functions do not feature in the expansion process hence the kinematics of the problem are unchanged and momentum conservation is preserved throughout. The final state integrations involve only stable on-shell particles and so things like complex scattering angles that have plagued certain other attempts at calculating such cross-sections do not arise. Problems near 'threshold' that have been noted elsewhere [10, 11] do not appear. This is as expected since unstable particles do not exhibit sharp thresholds as do stable ones. Branch cuts corresponding to their production lie off the real axis and such threshold problems must be calculational artifacts. There is no ambiguity or flexibility at any stage of the procedure. Because matrix elements are always evaluated with external stable states they remain gauge-invariant throughout. As guaranteed by Fredholm theory, there will be an exact factorization between the final state decay products of the unstable particle and the rest of the matrix element. Because of this exact factorization, the matrix element takes the form

$$\sigma(s) = \int_0^s dp_1^2 \int_0^{(\sqrt{s}-\sqrt{p_1^2})^2} dp_2^2 \sigma(s;p_1^2,p_2^2)\ \rho(p_1^2)\ \rho(p_2^2), \qquad (3)$$

where $\sigma(s;p_1^2,p_2^2)$ is the cross-section obtained from the resonant part of the matrix element alone and

$$\rho(p^2) \approx \frac{1}{\pi}.\frac{p^2(\Gamma_Z/M_Z)}{(p^2-M_Z^2)^2+\Gamma_Z^2 M_Z^2}\theta(p_0)\theta(p^2)$$

is Breit-Wigner-like convolution function.

In ref.[12] the process $e^+e^- \to \bar{\nu}_e e^- W^+ \to \bar{\nu}_e e^- u\bar{d}$ is treated as an example of how to include width-effects in matrix elements involving unstable particles. Much is made of electromagnetic $U(1)_{\text{e.m.}}$ gauge invariance and the possibility that the large mass ratio s/m_e^2 present in this process might amplify uncancelled gauge-dependence to a disastrous level [13, 14]. In particular it is shown that gauge invariance guarantees that the cross-section behaves as $\sim q^{-2}$ as $q^2 \to 0$ and not $\sim q^{-4}$ as it might otherwise do. However the question of how to simultaneously maintain $SU(2)_L$ and $U(1)_{\text{e.m.}}$ gauge invariance in calculations that take account of W boson width effects is not considered. The approach advocated is to include a subset of higher-order corrections to restore $U(1)_{\text{e.m.}}$ gauge invariance as was done in ref.s[15, 16]. This procedure is however inconvenient, difficult to apply consistently and adds considerably to the calculational labour involved. As noted in ref.[12] the reparation scheme is arbitrary.

It may be commented that the use of next order diagrams to fix problems at a given order for processes involving unstable particles has been noted elsewhere [17]. In that case troublesome imaginary parts of counterterms were observed to be canceled by next order diagrams.

2   Gauge Invariance for Massless Particles

When massless gauge particles are present special problems arise. Suppose we were to write a Green's function G^μ with a photon leg in terms of some basis of standard covariants, l_i^μ, and Lorentz scalars, A_i, in a form analogous to eq.(1) and (2),

$$G^\mu(p_1, p_2, ...) = \sum_i l_i^\mu(p_1, p_2, ...) A_i(p_1^2, p_2^2, ...).$$ (4)

The l_i^μ, of course, must transform in the same way that G^μ does.

The gauge condition on Green's functions $q_\mu G^\mu = 0$ means that the A_i are not linearly independent since they satisfy the condition $\sum_i (q \cdot l_i) A_i = 0$ where q_μ is the momentum of the photon leg.

Gauge-invariant Green's functions can be obtained using the methods of Bardeen and Tung [9]. Suppose that we have some exact Green's function expressed in terms of standard covariants and Lorentz scalars as in eq.(4). Since $q_\mu G^\mu = 0$ the Green's function is invariant under of the action of the operator

$$I_\nu^\mu = g_\nu^\mu - \frac{p^\mu q_\nu}{p \cdot q}$$ (5)

where p is some conveniently chosen momentum. The spinor-Lorentz tensor structure of the Green's function is spanned by the set $\{I_\nu^\mu l_i^\nu\}$. That is

$$G^\mu(p_1, p_2, ...) = \sum_i \left(I_\nu^\mu l_i^\nu(p_1, p_2, ...) \right) A_i(p_1^2, p_2^2, ...).$$ (6)

For an approximate Green's function that does not satisfy the $U(1)_{\text{e.m.}}$ gauge-condition, the operator I_ν^μ serves as a projection operator onto $\{I_\nu^\mu l_i^\nu\}$. From the $\{I_\nu^\mu l_i^\nu\}$ it is always possible to construct a new basis that is free of kinematic singularities that might have occurred when, for example, $(p \cdot q) = 0$.

In general consider a Green's function G^μ calculated using some incomplete expansion up to some order. It will consist of two parts

$$G^\mu = G_0^\mu + G_1^\mu$$ (7)

where G_0^μ is a consistent gauge-invariant contribution correct to given order of the calculation. G_1^μ is a spurious higher-order gauge-dependent correction. Because G_0^μ satisfies the gauge condition, $q_\mu G_0^\mu = 0$, it is invariant under the action of the operator I_μ^ν. Thus at a given order we may make the replacement

$$G^\mu \to I_\nu^\mu G^\nu = G_0^\mu + I_\nu^\mu G_1^\nu.$$

Only G_1^μ is affected by the projection operator but since it is of higher order this is of no concern. The important point is that it now satisfies the gauge condition $q_\mu(I_\nu^\mu G_1^\nu) = 0$ and is prevented from causing too strong a numerical blow up as $q^2 \to 0$.

3 The Process $e^+e^- \to \bar{\nu}_e e^- W^+ \to \bar{\nu}_e e^- u\bar{d}$

Consider the process $e^+e^- \to \bar{\nu}_e e^- W^+ \to \bar{\nu}_e e^- u\bar{d}$. Some of the tree level diagrams contributing to this process are shown in Fig.1. In the kinematic regions where the process is allowed (but W^+W^- production is forbidden) it is expected to be dominant because of the presence of a resonant W propagator. We will mainly concern ourselves with the region with q^2 small but the methods used apply in any kinematic region and will yield results that are exactly $SU(2)_L$ and $U(1)_{\text{e.m.}}$ gauge-invariant order by order.

Let q be the 4-momentum of the virtual photon and p_+ be that of the W^+. The momenta of the incoming electron and positron are p_1 and and k_1 respectively and p_2 and k_2 are those of the outgoing electron and $\bar{\nu}_e$. Hence

$$p_+ = p_u + p_d, \quad p_- = k_1 - k_2, \quad q = p_1 - p_2. \tag{8}$$

M_W is the W mass, e and g are the electromagnetic and $SU(2)_L$ weak coupling constants respectively. Q_i is the electric charge of particle i. As usual $\gamma_L = \frac{1}{2}(1 - \gamma_5)$ is the left-handed helicity operator.
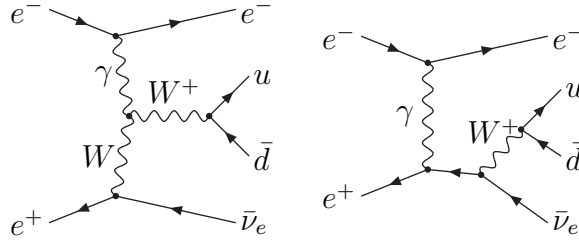

Figure 1: Tree-level diagrams contributing to the resonant part of the process $e^+e^- \to \bar{\nu}_e e^- W^+ \to \bar{\nu}_e e^- u\bar{d}$.

Using the same notation as in the introduction the exact matrix element takes the form

$$\mathcal{M} = \sum_i l_i(q, p_+, ...) A_i(q^2, p_+^2, ...) \tag{9}$$

This matrix element may be divided into a part that has a simple pole at $q^2 = 0$ and

a part that has no pole

$$\mathcal{M} = \sum_i l_i(q, p_+, ...) \frac{R_i(0, p_+^2, ...)}{q^2} + \sum_i l_i(q, p_+, ...) \frac{R_i(q^2, p_+^2, ...) - R_i(0, p_+^2, ...)}{q^2}. \quad (10)$$

These terms must be separately $SU(2)_L$ gauge-invariant because their differing pole structures. The first term receives contributions from diagrams, such as those of Fig.1, in which a photon is connected to the external electron current. The second term receives contributions from these as well many other topologies. Examples are the diagrams of Fig.1 in which the photon is replaced by a Z^0 boson.

We will be interested in the small q^2 region and will thus concentrate on the first term in eq.(10) and write

$$\mathcal{M} = \sum_i J_\mu^\gamma (l_i')^\mu (q, p_+, ...) R_i'(0, p_+^2, ...) \quad (11)$$

where

$$J_\mu^\gamma = \frac{V_\gamma(0)}{[1 - \Pi'_{\gamma\gamma}(0)]^{\frac{1}{2}}} \cdot \frac{\bar{u}(p_2)\gamma_\mu u(p_1)}{q^2}. \quad (12)$$

The $(l_i')^\mu (q, p_+, ...)$ are standard covariants with attached external wavefunctions and $V_\gamma(q^2)$ is the $\gamma e^+ e^-$ vertex form-factor. $\Pi'_{\gamma\gamma}(q^2)$ is the derivative of the photon self-energy with respect to q^2. The factorization of the matrix element in this way is guaranteed by Fredholm theory. Since $\mathcal{M}$ is $U(1)_{\text{e.m.}}$ gauge invariant it is possible to find a set $\{(l_i')^\mu (q, p_+, ...)\}$ that is free of kinematic singularities and that satisfy $q_\mu (l_i')^\mu (q, p_+, ...) = 0$ [9].

¿From the part of the matrix element (11) we wish to extract the dominant resonant part of the matrix element that describes the production of a physical W^+ and its subsequent decay into the $\bar{u}d$ final state. As with the process $e^+ e^- \to Z^0 Z^0$ this is the leading term in a Laurent expansion of $R_i(0, p_+^2, ...)$ in p_+^2, the invariant mass of the W^+ decay products. The result takes the form

$$\mathcal{M} = \sum_i J_\mu^\gamma J_\alpha^W (l_i'')^{\mu\alpha} (q, p_+, ...) R_i''(0, s_W, ...) \quad (13)$$

where

$$J_\alpha^W = \frac{V_W(s_W)}{[1 - \Pi'_{WW}(s_W)]^{\frac{1}{2}}} \cdot \frac{\bar{u}(p_u)\gamma_\alpha \gamma_L v(p_d)}{p_+^2 - s_W} \quad (14)$$

Here $\Pi'_{WW}(q^2)$ is the derivative of the W self energy with respect to q^2 and s_W the solution of $p_+^2 - M_W^2 - \Pi_{WW}(p_+^2) = 0$. $V_W(p_+^2)$ is $W^+ u\bar{d}$ vertex form factor. The result (13) is exactly $SU(2)_L$ and $U(1)_{\text{e.m.}}$ gauge-invariant.

Since the $(l_i'')^{\mu\alpha} (q, p_+, ...)$ are $U(1)_{\text{e.m.}}$ gauge-invariant they are invariant under the action of projection operators of the form given in eq.(5).

In finite order calculations, because of the non-perturbative nature of the W^+ resonance, $U(1)_{\text{e.m.}}$ gauge-invariance may not be satisfied. A gauge-invariant result

may be obtained by inserting a projection operator, I^μ_ν, as defined in eq.(5). This has the the effect of discarding spurious gauge-dependent terms that must ultimately cancel in the final result thus avoiding the need to calculate higher-order diagrams as was done in ref.s[12, 15].

With these general considerations in mind we can now calculate the cross-section for the process $\sigma(e^+e^- \to \bar{\nu}_e e^- W^+ \to \bar{\nu}_e e^- u\bar{d})$ in leading order and with the W width effects correctly taken into account. In words this is the production cross-section for $e^-\bar{\nu}_e$ and a physical W^+ that subsequently decays into a $u\bar{d}$ pair.

As discussed above the contribution relevant as $q^2 \to 0$ takes the form

$$\mathcal{M} = \mathcal{M}^{\mu\alpha} J^\gamma_\mu J^W_\alpha. \tag{15}$$

in which

$$\mathcal{M}^{\mu\alpha} = \sum_{i=1}^{2} \mathcal{M}^{\mu\alpha}_i. \tag{16}$$

The $\mathcal{M}^{\mu\alpha}_i$ correspond to the two diagrams of Fig.1,

$$\mathcal{M}^{\mu\alpha}_1 = ie\frac{g}{\sqrt{2}}Q_W\frac{1}{p^2_- - M^2_W}V^{\alpha\beta\mu}(p_+, -p_-, -q)\bar{v}(k_1)\gamma_\beta\gamma_L v(k_2), \tag{17}$$

$$\mathcal{M}^{\mu\alpha}_2 = ie\frac{g}{\sqrt{2}}Q_e\bar{v}(k_1)\gamma^\mu\frac{\slashed{k}_1 + \slashed{q}}{(k_1 + q)^2}\gamma^\alpha\gamma_L v(k_2). \tag{18}$$

$$\tag{19}$$

and

$$V^{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = (p_1 - p_2)^{\mu_3}g^{\mu_1\mu_2} + (p_2 - p_3)^{\mu_1}g^{\mu_2\mu_3} + (p_3 - p_1)^{\mu_2}g^{\mu_3\mu_1}. \tag{20}$$

The projection operator I^μ_ν of eq.(5) may be inserted into eq.(15) in order to guarantee $U(1)_{e.m.}$ gauge invariance of the Green's function, $\mathcal{M}^{\mu\alpha}$. The matrix element then becomes

$$\mathcal{M} = \mathcal{M}^{\mu\alpha} I^\lambda_\mu J^\gamma_\lambda J^W_\alpha. \tag{21}$$

The matrix element squared and averaged over spins then takes the form

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{4}\sum_{e^+e^- \text{ spins}} \mathcal{M}^{\mu\alpha}I^\lambda_\mu J^\gamma_\lambda g_{\alpha\beta}\rho(p^2_+)\overline{\mathcal{M}^{\nu\beta}I^\rho_\nu J^\gamma_\rho} \tag{22}$$

with

$$g_{\alpha\beta}\rho(p^2_+) = \sum_{u\bar{d} \text{ spins}} \int \frac{d^3p_u}{(2\pi)^3 2p^0_u}\frac{d^3p_d}{(2\pi)^3 2p^0_d}J^W_\alpha\overline{J^W_\beta}(2\pi)^4\delta^4(p_u + p_d - p_+)\theta(p^0_+)\theta(p^2_+)$$

$$= \frac{g^2}{48\pi^2}\cdot\frac{p^2_+}{|p^2_+ - s_W|^2}g_{\alpha\beta}\theta(p^0_+)\theta(p^2_+) \tag{23}$$

at leading order.

7

In forming the total production cross-section for a physical W^+'s that subsequently decays into $u\bar{d}$ we must sum over the $u\bar{d}$ final states. Hence

$$\sigma(e^+e^- \to \bar{\nu}_e e^- W^+ \to \bar{\nu}_e e^- u\bar{d}) = N_c \int dp^2 \sigma(p^2)\rho(p^2). \tag{24}$$

The integral is over the full range allowed by the cut on the scattering angle, θ. N_c is the number of QCD colours and

$$\begin{aligned}
\sigma(p^2) &= \frac{1}{2s} \int \frac{d^3p_2}{(2\pi)^3 2p_2^0} \frac{d^3k_2}{(2\pi)^3 2k_2^0} \frac{d^3p_+}{(2\pi)^3 2p_+^0} \left\langle |\mathcal{M}_0|^2 \right\rangle \\
&\quad \times (2\pi)^4 \delta^4(p_1 + k_1 - p_2 - k_2 - p_+)
\end{aligned} \tag{25}$$

in which

$$\left\langle |\mathcal{M}_0|^2 \right\rangle = \frac{1}{4} \sum_{e^+e^- \text{ spins}} \mathcal{M}^{\mu\alpha} I_\mu^\lambda J_\lambda^\gamma \overline{\mathcal{M}^{\nu\alpha} I_\nu^\rho J_\rho^\gamma} \tag{26}$$

and the integral is evaluated under the constraint that $p_+^2 = p^2$. Because $\mathcal{M}^{\mu\alpha} I_\mu^\lambda$ is $U(1)_{\text{e.m.}}$ gauge-invariant, $\left\langle |\mathcal{M}_0|^2 \right\rangle \sim q^{-2}$ only as $q^2 \to 0$.

4   Acknowledgments

The author wishes to thank D. Williams for many enlightening discussions and the authors of ref.[12] for permitting use of their figures of Feynman diagrams.

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